

# Some congruences involving powers of Delannoy polynomials

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**Abstract.** The Delannoy polynomial  $D_n(x)$  is defined by

$$D_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k.$$

We prove that, if  $x$  is an integer and  $p$  is a prime not dividing  $x(x+1)$ , then

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1) D_k(x)^3 &\equiv p \left( \frac{-4x-3}{p} \right) \pmod{p^2}, \\ \sum_{k=0}^{p-1} (2k+1) D_k(x)^4 &\equiv p \pmod{p^2}, \\ \sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^3 &\equiv p \left( \frac{4x+1}{p} \right) \pmod{p^2}, \end{aligned}$$

where  $\left(\frac{\cdot}{p}\right)$  denotes the Legendre symbol. The first two congruences confirm a conjecture of Z.-W. Sun [Sci. China 57 (2014), 1375–1400]. The third congruence confirms a special case of another conjecture of Z.-W. Sun [J. Number Theory 132 (2012), 2673–2699]. We also prove that, for any integer  $x$  and odd prime  $p$ , there holds

$$\sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^4 \equiv p \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k}^2 (x^2+x)^k (2x+1)^{2k} \pmod{p^2},$$

and conjecture that it holds modulo  $p^3$ .

*Keywords:* congruences; Delannoy polynomials; Clausen's formula; Zeilberger's algorithm; Fermat's little theorem

*MR Subject Classifications:* 11A07, 11B65, 05A10

# 1 Introduction

The central Delannoy numbers (see [1, 9]) are defined by

$$D_n = \sum_{k=0}^n \binom{n+k}{2k} \binom{2k}{k}.$$

Z.-W. Sun [11–13], among other things, proved many interesting congruences on sums involving Delannoy numbers, such as

$$\begin{aligned} \sum_{k=0}^{p-1} (2k+1)D_k &\equiv p + 2p(2^{p-1} - 1) - p(2^{p-1} - 1)^2 \pmod{p^4}, \\ \sum_{k=0}^{n-1} (2k+1)D_k^2 &\equiv 0 \pmod{n^2}, \end{aligned}$$

where  $p$  is a prime greater than 3. Z.-W. Sun [13] also introduced the Delannoy polynomial  $D_n(x)$  as follows:

$$D_n(x) = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} x^k,$$

i.e.,  $D_n(x) = P_n(2x+1)$ , where  $P_n(x)$  is the Legendre polynomial of degree  $n$  (see, for example, [6, p. 1]). Then he raised the following conjecture.

**Conjecture 1.1** [13, Conjecture 5.1] *Let  $x$  be an integer and let  $m$  and  $n$  be positive integers. Then*

$$\sum_{k=0}^{n-1} (2k+1)D_k(x)^m \equiv 0 \pmod{n}. \quad (1.1)$$

*If  $p$  is a prime not dividing  $x(x+1)$ , then*

$$\sum_{k=0}^{p-1} (2k+1)D_k(x)^3 \equiv p \left( \frac{-4x-3}{p} \right) \pmod{p^2}, \quad (1.2)$$

$$\sum_{k=0}^{p-1} (2k+1)D_k(x)^4 \equiv p \pmod{p^2}, \quad (1.3)$$

where  $\left(\frac{\cdot}{p}\right)$  denotes the Legendre symbol.

The congruence (1.1) in a more general form has been confirmed by Pan [7] recently. However, Pan [7] did not give an integer coefficient polynomial formula for

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k+1)D_k(x)^m.$$

In this paper we shall prove the following results.

**Theorem 1.2** *Let  $n$  be a positive integer. Then*

$$\begin{aligned} & \frac{1}{n} \sum_{k=0}^{n-1} (2k+1) D_k(x)^3 \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^i \binom{n}{j+k+1} \binom{n+j+k}{j+k} \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{2i}{i} x^{i+j} (x+1)^i, \end{aligned} \quad (1.4)$$

$$\begin{aligned} & \frac{1}{n} \sum_{k=0}^{n-1} (2k+1) D_k(x)^4 \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^i \binom{n}{j+k+1} \binom{n+j+k}{j+k} \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{2i}{i} \binom{2j}{j} (x^2+x)^{i+j}. \end{aligned} \quad (1.5)$$

**Theorem 1.3** *The supercongruences (1.2) and (1.3) are true.*

**Theorem 1.4** *Let  $x$  be an integer and  $p$  an odd prime. Then*

$$\sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^3 \equiv p \left( \frac{4x+1}{p} \right) \pmod{p^2}, \text{ provided that } p \nmid x(x+1), \quad (1.6)$$

$$\sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^4 \equiv p \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k}^2 (x^2+x)^k (2x+1)^{2k} \pmod{p^2}. \quad (1.7)$$

For any positive integer  $n$  and  $p$ -adic integer  $x$ , Z.-W. Sun [12, (4.6)] conjectured that

$$\nu_p \left( \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k+1) D_k(x)^3 \right) \geq \min\{\nu_p(n), \nu_p(4x+1)\}, \quad (1.8)$$

where  $\nu_p(x)$  denotes the  $p$ -adic valuation of  $x$ . It is clear that the congruence (1.6) confirms the  $n = p$  case of (1.8).

## 2 Proof of Theorem 1.2

It is easy to see that (see [10, Lemma 3.2])

$$D_n(x)^2 = \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} x^k (x+1)^k, \quad (2.1)$$

which can be deduced from Clausen's formula [3] (with  $a = -\frac{n}{2}$ ,  $b = \frac{n+1}{2}$  and  $x \rightarrow -4x(x+1)$ ):

$${}_2F_1 \left[ \begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; x \right]^2 = {}_3F_2 \left[ \begin{matrix} 2a, 2b, a+b \\ 2a+2b, a+b+\frac{1}{2} \end{matrix}; x \right], \quad |x| < 1, \quad (2.2)$$

and the following quadratic transformation of Gauss hypergeometric function (see [6, p. 180]):

$${}_2F_1 \left[ \begin{matrix} a, b \\ a+b+\frac{1}{2} \end{matrix}; 4x(1-x) \right] = {}_2F_1 \left[ \begin{matrix} 2a, 2b \\ a+b+\frac{1}{2} \end{matrix}; x \right]. \quad (2.3)$$

Writing  $D_\ell(x)^3$  as  $D_\ell(x)^2 \cdot D_\ell(x)$  and applying (2.1), we have

$$\begin{aligned} & \frac{1}{n} \sum_{\ell=0}^{n-1} (2\ell+1) D_\ell(x)^3 \\ &= \frac{1}{n} \sum_{\ell=0}^{n-1} (2\ell+1) \sum_{i=0}^{\ell} \binom{\ell}{i} \binom{\ell+i}{i} \binom{2i}{i} x^i (x+1)^i \sum_{j=0}^{\ell} \binom{\ell}{j} \binom{\ell+j}{j} x^j. \end{aligned} \quad (2.4)$$

Note that (see the proof of [5, Lemma 4.2])

$$\binom{\ell}{i} \binom{\ell+i}{i} \binom{\ell}{j} \binom{\ell+j}{j} = \sum_{k=0}^i \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{\ell}{j+k} \binom{\ell+j+k}{j+k}. \quad (2.5)$$

Moreover, by induction on  $n$ , we can easily prove that

$$\sum_{\ell=k}^{n-1} (2\ell+1) \binom{\ell}{k} \binom{\ell+k}{k} = n \binom{n}{k+1} \binom{n+k}{k}. \quad (2.6)$$

Substituting (2.5) into (2.4), exchanging the summation order, and then utilizing (2.6), we complete the proof of (1.4).

Similarly, writing  $D_\ell(x)^4$  as  $D_\ell(x)^2 \cdot D_\ell(x)^2$  and applying (2.1), we can prove (1.5).

### 3 Proof of Theorem 1.3

*Proof of (1.2).* Letting  $n = p$  be a prime in (1.4), and noticing that  $\binom{p}{k} \equiv 0 \pmod{p}$  for  $1 \leq k \leq p-1$  and  $\binom{2p-1}{p} \equiv 1 \pmod{p}$ , we obtain

$$\begin{aligned} & \frac{1}{p} \sum_{k=0}^{p-1} (2k+1) D_k(x)^3 \\ &= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^i \binom{p}{j+k+1} \binom{p+j+k}{j+k} \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{2i}{i} x^{i+j} (x+1)^i \\ &\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{i+j}{i} \binom{j}{p-i-1} \binom{p-1}{j} \binom{2i}{i} x^{i+j} (x+1)^i \pmod{p}. \end{aligned} \quad (3.1)$$

For  $0 \leq i, j \leq p-1$ , there holds

$$\binom{i+j}{i} \binom{j}{p-i-1} \begin{cases} = 0, & \text{if } i+j < p-1, \\ \equiv 0 \pmod{p}, & \text{if } i+j \geq p. \end{cases} \quad (3.2)$$

Therefore, the possible nonzero summands in (3.1) must satisfy  $i+j = p-1$ . In other words, the congruence (3.1) may be simplified as

$$\begin{aligned} \frac{1}{p} \sum_{k=0}^{p-1} (2k+1) D_k(x)^3 &\equiv \sum_{i=0}^{p-1} \binom{p-1}{i} \binom{p-1}{p-i-1} \binom{2i}{i} x^{p-1} (x+1)^i \\ &\equiv \sum_{i=0}^{p-1} \binom{2i}{i} (x+1)^i \pmod{p}, \end{aligned}$$

where we used the fact  $\binom{p-1}{i} \equiv (-1)^i \pmod{p}$  and Fermat's little theorem. The proof then follows from the congruence

$$\sum_{k=0}^{p-1} \binom{2k}{k} x^k \equiv \left( \frac{1-4x}{p} \right) \pmod{p} \quad (3.3)$$

due to Sun and Tauraso [14, Theorem 1.1] (see also [13, Lemma 2.1]).  $\square$

*Proof of (1.3).* Let  $n = p$  be a prime in (1.5). Similarly to the proof of (1.2), we have

$$\begin{aligned} \frac{1}{p} \sum_{k=0}^{p-1} (2k+1) D_k(x)^4 &\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{i+j}{i} \binom{j}{p-i-1} \binom{p-1}{j} \binom{2i}{i} \binom{2j}{j} (x^2+x)^{i+j} \\ &\equiv \sum_{i=0}^{p-1} \binom{2i}{i} \binom{2p-2i-2}{p-i-1} \pmod{p} \quad (\text{by (3.2) and Fermat's little theorem}) \\ &\equiv 1 \pmod{p}, \end{aligned}$$

where in the last step we used the following fact

$$\binom{2i}{i} \equiv 0 \pmod{p} \quad \text{for } \frac{p-1}{2} < i < p, \quad (3.4)$$

and  $\binom{p-1}{\frac{p-1}{2}}^2 \equiv 1 \pmod{p}$ .  $\square$

## 4 Proof of Theorem 1.4

We need the following two lemmas.

**Lemma 4.1** *Let  $n$  be a positive integer. Then*

$$\begin{aligned} & \frac{1}{n} \sum_{k=0}^{n-1} (-1)^{n-k-1} (2k+1) D_k(x)^3 \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^i \binom{n-1}{j+k} \binom{n+j+k}{j+k} \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{2i}{i} x^{i+j} (x+1)^i, \end{aligned} \quad (4.1)$$

$$\begin{aligned} & \frac{1}{n} \sum_{k=0}^{n-1} (-1)^{n-k-1} (2k+1) D_k(x)^4 \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^i \binom{n-1}{j+k} \binom{n+j+k}{j+k} \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{2i}{i} \binom{2j}{j} (x^2+x)^{i+j}. \end{aligned} \quad (4.2)$$

*Proof.* It is exactly similar to the proof of Theorem 1.2. The difference is that we need to replace (2.6) by the following identity:

$$\sum_{\ell=k}^{n-1} (-1)^{n-\ell-1} (2\ell+1) \binom{\ell}{k} \binom{\ell+k}{k} = n \binom{n-1}{k} \binom{n+k}{k},$$

which can also be proved by induction on  $n$ . □

**Lemma 4.2** *Let  $n$  be a positive integer. Then*

$$\sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k} = \sum_{k=0}^n \binom{2k}{k}^2 \binom{k}{n-k} (-4)^{n-k}. \quad (4.3)$$

*Proof.* Applying Zeilberger's algorithm (see [6, 8]), we find that both sides of (4.3) satisfy the following recurrence relation:

$$(n+2)^2 S(n+2) - 4(3n^2 + 9n + 7) S(n+1) + 32(n+1)^2 S(n) = 0.$$

Noticing that they also have the same initial values  $S(0) = 1$  and  $S(1) = 4$ , we complete the proof. □

*Proof of (1.6).* Letting  $n = p$  be a prime not dividing  $x(x+1)$  in (4.1), we have

$$\begin{aligned}
& \frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^3 \\
&= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^i \binom{p-1}{j+k} \binom{p+j+k}{j+k} \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{2i}{i} x^{i+j} (x+1)^i \\
&\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^i (-1)^{j+k} \binom{i+j}{i} \binom{j}{i-k} \binom{j+k}{k} \binom{2i}{i} x^{i+j} (x+1)^i \pmod{p}, \quad (4.4)
\end{aligned}$$

where we used the fact that, for  $0 \leq j, k \leq p-1$ ,

$$\binom{p-1}{j+k} \binom{p+j+k}{j+k} \binom{j+k}{k} \equiv (-1)^{j+k} \binom{j+k}{k} \pmod{p}.$$

By the Chu-Vandermonde summation formula, we get

$$\sum_{k=0}^i (-1)^k \binom{j}{i-k} \binom{j+k}{k} = (-1)^i. \quad (4.5)$$

Substituting (4.5) into (4.4) and using the binomial theorem, we obtain

$$\begin{aligned}
& \frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^3 \\
&\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^{i+j} \binom{i+j}{i} \binom{2i}{i} x^{i+j} (x+1)^i \\
&= \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \sum_{k=0}^i (-1)^{i+j} \binom{i+j}{i} \binom{2i}{i} \binom{i}{k} x^{i+j+k} \\
&\equiv \sum_{m=0}^{3p-3} x^m \sum_{i=0}^{\min\{p-1, m\}} (-1)^i \binom{2i}{i} \sum_{j=0}^{\min\{p-1, m-i\}} (-1)^j \binom{i+j}{i} \binom{i}{m-i-j} \pmod{p}. \quad (4.6)
\end{aligned}$$

Note that, if  $m-i \leq p-1$ , then

$$\sum_{j=0}^{\min\{p-1, m-i\}} (-1)^j \binom{i+j}{i} \binom{i}{m-i-j} = \sum_{j=0}^{m-i} (-1)^j \binom{i+j}{i} \binom{i}{m-i-j} = (-1)^{m-i};$$

while if  $m-i \geq p$ , then for  $0 \leq i, j \leq p-1$ , there holds  $\binom{i+j}{i} \binom{i}{m-i-j} \equiv 0 \pmod{p}$ . Hence, we may simplify (4.6) to

$$\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^3 \equiv \sum_{m=0}^{p-1} (-x)^m \sum_{i=0}^m \binom{2i}{i} + \sum_{m=p}^{2p-2} (-x)^m \sum_{i=m-p+1}^{p-1} \binom{2i}{i} \pmod{p}. \quad (4.7)$$

By (3.3) and Fermat's little theorem, we have

$$\begin{aligned}
\sum_{m=0}^{p-1} (-x)^m \sum_{i=0}^m \binom{2i}{i} &= \sum_{i=0}^{p-1} \binom{2i}{i} \sum_{m=i}^{p-1} (-x)^m \\
&= \sum_{i=0}^{p-1} \binom{2i}{i} \frac{(-x)^i - (-x)^p}{1+x} \\
&\equiv \frac{1}{1+x} \left( \frac{1+4x}{p} \right) + \frac{x}{1+x} \left( \frac{-3}{p} \right) \pmod{p}, \tag{4.8}
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=p}^{2p-2} (-x)^m \sum_{i=m-p+1}^{p-1} \binom{2i}{i} &= (-x)^p \sum_{m=0}^{p-2} (-x)^m \sum_{i=m+1}^{p-1} \binom{2i}{i} \\
&= (-x)^p \sum_{i=1}^{p-1} \binom{2i}{i} \sum_{m=0}^{i-1} (-x)^m \\
&= (-x)^p \sum_{i=0}^{p-1} \binom{2i}{i} \frac{1 - (-x)^i}{1+x} \\
&\equiv \frac{-x}{1+x} \left( \frac{-3}{p} \right) + \frac{x}{1+x} \left( \frac{1+4x}{p} \right) \pmod{p}, \tag{4.9}
\end{aligned}$$

Substituting (4.8) and (4.9) into (4.7), we complete the proof.  $\square$

*Proof of (1.7).* Let  $n = p$  be a prime in (4.2). Then similarly to (4.4) we have

$$\begin{aligned}
\frac{1}{p} \sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^4 &\equiv \sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (-1)^{i+j} \binom{i+j}{i} \binom{2i}{i} \binom{2j}{j} (x^2+x)^{i+j} \\
&= \sum_{n=0}^{p-1} (-1)^n (x^2+x)^n \sum_{i=0}^n \binom{n}{i} \binom{2i}{i} \binom{2n-2i}{n-i} \pmod{p}, \tag{4.10}
\end{aligned}$$

where we used the fact that  $\binom{i+j}{i} \equiv 0 \pmod{p}$  for  $0 \leq i, j \leq p-1$  and  $i+j \geq p$ .



By (4.3), the right-hand side of (4.10) is equal to

$$\begin{aligned}
& \sum_{n=0}^{p-1} (-1)^n (x^2 + x)^n \sum_{k=0}^n \binom{2k}{k}^2 \binom{k}{n-k} (-4)^{n-k} \\
&= \sum_{k=0}^{p-1} (-1)^k \binom{2k}{k}^2 \sum_{n=k}^{p-1} \binom{k}{n-k} 4^{n-k} (x^2 + x)^n \\
&\equiv \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k}^2 \sum_{n=k}^{2k} \binom{k}{n-k} 4^{n-k} (x^2 + x)^n \\
&= \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k}^2 (x^2 + x)^k (1 + 4x + 4x^2)^k \pmod{p},
\end{aligned}$$

where we used the congruence (3.4) and the binomial theorem. This completes the proof.  $\square$

## 5 Two open problems

Motivated by (1.8), we raise the following conjecture:

**Conjecture 5.1** *Let  $n$  be a positive integer and  $x$  a  $p$ -adic integer. Then*

$$\nu_p \left( \frac{1}{n} \sum_{k=0}^{n-1} (2k+1) D_k(x)^3 \right) \geq \min\{\nu_p(n), \nu_p(4x+3)\}. \quad (5.1)$$

It is obvious that Theorem 1.3 means that the  $n = p$  case of (5.1) is true.

Finally, numerical calculation suggests the following refinement of (1.7).

**Conjecture 5.2** *Let  $x$  be an integer and  $p$  an odd prime. Then*

$$\sum_{k=0}^{p-1} (-1)^k (2k+1) D_k(x)^4 \equiv p \sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k}^2 (x^2 + x)^k (2x+1)^{2k} \pmod{p^3}.$$

**Acknowledgments.** This work was partially supported by the Fundamental Research Funds for the Central Universities and the National Natural Science Foundation of China (grant 11371144).

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